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ε-uniform schemes with high-order time-accuracy for parabolic singular perturbation problems

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In this paper we study the discrete approximation of a Dirichlet problem on an interval for a singularly perturbed parabolic PDE. The highest derivative in the equation is multiplied by an arbitrarily small parameter ε. If the parameter vanishes, the parabolic equation degenerates to a first-order equation, in which only the time derivative remains. For small values of the parameter, boundary layers may appear that give rise to difficulties when classical discretization methods are applied. Then the error in the approximate solution depends on the value of ε. An adapted placement of the nodes is needed to ensure that the error is independent of the parameter value and depends only on the number of nodes in the mesh. Special schemes with this property are called ε-uniformly convergent. In this paper such ε-uniformly convergent schemes are studied, which combine a difference scheme and a mesh selection criterion for the space discretization.

Except for a small logarithmic factor, the order of convergence is one and two with respect to the time and space variables, respectively. Therefore, it is of interest to develop methods for which the order of convergence with respect to the time variable is increased. In this paper we develop schemes for which the order of convergence in time can be arbitrarily large if the solution is sufficiently smooth. To obtain uniform convergence, we use a mesh with nodes that are condensed in the neighbourhood of the boundary layers. To obtain a better approximation in time, we use auxiliary discrete problems on the same time-mesh to correct the difference approximations. In this sense, the present algorithm is an improvement over a previously published one. To validate the theoretical results, some numerical results for the new schemes are presented.

Keywords: parabolic PDEs; higher-order time-accuracy schemes; defect correction; ε-uniform convergence.

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1. Introduction

In this paper we study ε-uniform schemes for time-dependent singular perturbation problems. For a general discussion of ε-uniform schemes for singular perturbation problems we refer to Doolan *et al.* (1980), Shishkin (1992), Hemker *et al.* (1997), Morton (1996), Roos *et al.* (1996). In earlier papers (Farrell *et al.*, 1996a, b, c; Hemker *et al.*, 1997) we have introduced and analysed ε-uniformly convergent difference schemes for singularly perturbed boundary value problems for elliptic and parabolic equations. If the problem data are sufficiently smooth, for the parabolic equations without convection terms, the order of ε-uniform convergence for the scheme that was studied is $\mathcal{O}(N^{-2}\ln^2 N + K^{-1})$, where N and K denote, respectively, the number of intervals in the space and time discretization. For this scheme the amount of computational work is primarily determined by the time discretization, which is of first-order accuracy only. The difficulty for the singular perturbation problem, however, lies essentially in the space direction where we have second-order accuracy. Therefore, it is natural to search for a method that has the same order of accuracy for both variables. To this end, we want to improve the accuracy with respect to the time step, without essentially increasing the amount of computational work. The improvement of the accuracy in time, maintaining ε-uniform convergence, by means of a defect correction technique was also studied in Hemker *et al.* (1997). In that paper, higher-order backward differences were used to obtain a better approximation of the time derivative. To determine the derivatives, finite difference schemes on a sequence of finer time-meshes were used. Therefore, the implementation of the schemes in Hemker *et al.* (1997) appeared somewhat cumbersome. In the present paper we develop a new approach, also based on the defect correction principle, but which is easier to implement and analyse, as it only uses a single time-mesh, which is the same for all auxiliary problems.

By this method we are able to achieve the same order of accuracy in both variables. Moreover, we present a method which can achieve a higher order of accuracy with respect to the time variable. Thus, the accuracy of this method is restricted essentially by the second-order accuracy in space, which is the natural limit set by the character of the problem.

2. The class of boundary value problems studied

On the domain $G = (0, 1) \times (0, T]$, with boundary $S = \overline{G} \setminus G$ we consider the following singularly perturbed parabolic equation with Dirichlet boundary conditions[†]:

$$L_{(2.1)}u(x,t) \equiv \varepsilon^2 \frac{\partial}{\partial x}\left(a(x,t)\frac{\partial u}{\partial x}(x,t)\right) - c(x,t)u(x,t) \tag{2.1a}$$

$$-p(x,t)\frac{\partial u}{\partial t}(x,t) = f(x,t), \qquad (x,t) \in G,$$

$$u(x,t) = \varphi(x,t), \qquad (x,t) \in S. \tag{2.1b}$$

For $S = S_0 \cup S_1$, we distinguish the lateral boundary $S_1 = \{(x,t) : x = 0 \text{ or } x = 1, 0 \leqslant t \leqslant T\}$, and the initial boundary $S_0 = \{(x,t) : x \in [0,1], t = 0\}$. In (2.1b), $a(x,t)$,

† The notation is such that the operator $L_{(a.b)}$ is first introduced in equation $(a.b)$.

$c(x,t)$, $p(x,t)$, $f(x,t)$, $(x,t) \in \overline{G}$, and $\varphi(x,t)$, $(x,t) \in S$ are sufficiently smooth and bounded functions which satisfy

$$0 < a_0 \leqslant a(x,t), \qquad 0 < p_0 \leqslant p(x,t), \qquad c(x,t) \geqslant 0, \qquad (x,t) \in \overline{G}. \tag{2.1c}$$

The real parameter ε may take any small positive value, say

$$\varepsilon \in (0, 1]. \tag{2.1d}$$

When the parameter ε tends to zero in (2.1a), layers appear in the solution in the neighbourhood of the lateral boundary, which are described by an equation of parabolic type (parabolic boundary layers). If an additional first-order term $b(x,t)(\partial u(x)/\partial x)$ was present in (2.1a) then we would see a boundary layer at the outflow boundary, that would be described by an ordinary differential equation (an ordinary boundary layer).

3. An arbitrary non-uniform mesh

To solve problem (2.1) we first consider a classical finite difference method on a (possibly) non-uniform mesh. On the set $\overline{G}$ we introduce the rectangular mesh

$$\overline{G}_h = \overline{\omega} \times \overline{\omega}_0, \tag{3.1}$$

where $\overline{\omega}$ is the (possibly) non-uniform mesh of nodal points, x^i, in [0, 1], $\overline{\omega}_0$ is a uniform mesh on the interval $[0, T]$; N and K are the numbers of intervals in the meshes $\overline{\omega}$ and $\overline{\omega}_0$ respectively. We define $\tau = T/K$, $h^i = x^{i+1} - x^i$, $h = \max_i h^i$, $h \leqslant M/N$, $G_h = G \cap \overline{G}_h$, $S_h = S \cap \overline{G}_h$.

Here and below we denote by M (or m) sufficiently large (or small) positive constants which do not depend on the value of the parameter ε or on the difference operators.

For problem (2.1) we use the difference scheme (Samarski, 1989)

$$\Lambda_{(3.2)} z(x,t) = f(x,t), \qquad (x,t) \in G_h, \tag{3.2a}$$

$$z(x,t) = \varphi(x,t), \qquad (x,t) \in S_h. \tag{3.2b}$$

Here

$$\Lambda_{(3.2)} z(x,t) \equiv \varepsilon^2 \delta_{\hat{x}} \left(a^h(x,t) \delta_{\overline{x}} z(x,t) \right) - c(x,t) z(x,t) - p(x,t) \delta_{\overline{t}} z(x,t),$$

$$\delta_{\hat{x}} \left(a^h(x^i,t)\, \delta_{\overline{x}} z(x^i,t) \right) = 2(h^{i-1} + h^i)^{-1} \left(a^h(x^{i+1},t) \delta_x z(x^i,t) - a^h(x^i,t) \delta_{\overline{x}} z(x^i,t) \right),$$

$$a^h(x^i,t) = a\left((x^{i-1} + x^i)/2, t \right),$$
$$\delta_{\overline{x}} z(x^i,t) = (h^{i-1})^{-1} \left(z(x^i,t) - z(x^{i-1},t) \right),$$
$$\delta_x z(x^i,t) = (h^i)^{-1} \left(z(x^{i+1},t) - z(x^i,t) \right),$$
$$\delta_{\overline{t}} z(x^i,t) = \tau^{-1} \left(z(x^i,t) - z(x^i,t-\tau) \right),$$

$\delta_x z(x,t)$ and $\delta_{\overline{x}} z(x,t)$, $\delta_{\overline{t}} z(x,t)$ are the forward and backward differences, and the difference operator and $\delta_{\hat{x}}(a^h(x,t)\delta_{\overline{x}} z(x,t))$ is an approximation of the operator $\frac{\partial}{\partial x}(a(x,t)\frac{\partial}{\partial x}u(x,t))$ on the non-uniform mesh.

From Samarski (1989) we know that the difference scheme (3.2), (3.1) is monotone. By means of the maximum principle and taking into account estimates of the derivatives (see Theorem 5 in the Appendix) we find that the solution of the difference scheme (3.2), (3.1) converges for a fixed value of the parameter ε:

$$|u(x,t) - z(x,t)| \leqslant M(\varepsilon^{-1}N^{-1} + \tau), \qquad (x,t) \in \overline{G}_h. \tag{3.3}$$

This error bound for the classical difference scheme is clearly not ε-uniform.

The proof of (3.3) follows the lines of the classical convergence proof for monotone difference schemes (cf. Samarski, 1989; Shishkin, 1992). Taking into account an *a priori* estimate for the solution (Appendix, Section A.1), this results in the following theorem.

THEOREM 1 Let the estimate (A.2) hold for the solution of (2.1). Then, for a fixed value of the parameter ε, the solution of (3.2), (3.1) converges to the solution of (2.1) with an error bound given by (3.3).

4. The ε-uniformly convergent method

In this section we discuss an ε-uniformly convergent method for (2.1) by taking a special mesh, condensed in the neighbourhood of the boundary layers. The location of the nodes is derived from *a priori* estimates of the solution and its derivatives. The way to construct the mesh for problem (2.1) is the same as in Shishkin (1992) and Hemker *et al.* (1997). Specifically, we take

$$\overline{G}_h^* = \overline{\omega}^*(\sigma) \times \overline{\omega}_0, \tag{4.1}$$

where $\overline{\omega}_0$ is the uniform mesh with step-size $\tau = T/K$, i.e. $\overline{\omega}_0 = \overline{\omega}_{0(3.1)}$, and $\overline{\omega}^* = \overline{\omega}^*(\sigma)$ is a special *piecewise* uniform mesh depending on the parameter $\sigma \in \mathbb{R}$, which depends on ε and N. We take $\sigma = \sigma_{(4.1)}(\varepsilon, N) = \min(1/4, m\varepsilon \ln N)$, where $m = m_{(4.1)}$ is an arbitrary positive number. The mesh $\overline{\omega}^*(\sigma)$ is constructed as follows. The interval $[0,1]$ is divided into three parts $[0,\sigma]$, $[\sigma, 1-\sigma]$, $[1-\sigma, 1]$, $0 < \sigma \leqslant 1/4$. In each part we use a uniform mesh, with $N/2$ subintervals in $[\sigma, 1-\sigma]$ and with $N/4$ subintervals in each interval $[0,\sigma]$ and $[1-\sigma, 1]$.

THEOREM 2 If the solution of problem (2.1) satisfies the conditions of Theorem 5 (Appendix), then the solution of (3.2), (4.1) converges ε-uniformly to the solution of (2.1) and the following estimate holds:

$$|u(x,t) - z(x,t)| \leqslant M(N^{-2}\ln^2 N + \tau), \qquad (x,t) \in \overline{G}_h^*. \tag{4.2}$$

The proof of this theorem can be found in Shishkin (1992).

5. Numerical results

To see the effect of the special mesh in practice, we take the model problem

$$L_{(5.1)}u(x,t) \equiv \varepsilon^2 \frac{\partial^2 u}{\partial x^2}(x,t) - \frac{\partial u}{\partial t}(x,t) = f(x,t), \qquad (x,t) \in G, \tag{5.1}$$

$$u(x,t) = \varphi(x,t), \qquad (x,t) \in S,$$

where

$$f(x,t) = 4t^3, \qquad (x,t) \in \overline{G}, \qquad \varphi(x,t) = 0, \qquad (x,t) \in S.$$

In Hemker *et al.* (1997) we compared the numerical results for scheme (3.2) on the uniform mesh $\overline{G}_h^u = \overline{G}_{h\,(3.1)}$, where $\overline{\omega} = \overline{\omega}^u$ is a uniform mesh, and on the special mesh (4.1), adapted to the value ε. For $T = 1$ we presented the error $E(N, K, \varepsilon)$, defined by

$$E(N, K, \varepsilon) = \max_{(x,t)\in\overline{G}_h} |z(x,t) - u^*(x,t)|. \tag{5.2}$$

Here $u^*(x,t)$ is the piecewise linear interpolation obtained from the numerical solution $z(x,t)$ on an adapted mesh with parameters $\sigma = \min(1/4, 2\varepsilon \ln N)$, $N = K = 512$. Notice that no special interpolation is needed for the time variable.

In Hemker *et al.* (1997) we compared results for the uniform and adapted mesh, and we showed the errors $E(N, N, \varepsilon)$, for $N = 2^k$, $K = 3, \ldots, 8$, for various values of ε. From the results in Hemker *et al.* (1997), ε-uniform convergence can be observed but it is difficult to analyse the order of ε-uniform convergence in space and in time. Therefore, here we want to supplement these numerical results with values for $E(N, N^2, \varepsilon)$, for the adapted mesh, for the same N and for the same values of ε.

In Table 1 we give the results for the same scheme (3.2), (4.1) but with $K = N^2$. Here we can clearly see that, in accordance with estimate (4.2), the order of convergence is $\mathcal{O}(N^{-2}\ln^2 N + K^{-1})$. For large N the order of convergence 2 (resp. 1) with respect to the space and time variable corresponds with the theoretical results.

TABLE 1

Errors $E(N, N^2, \varepsilon)$ for the special method (3.2), (4.1). In this table the function $E(N, N^2, \varepsilon)$ is defined by (5.2), but now $z(x,t)$ in (5.2) is the solution of (3.2), (4.1) with $m = 2$, $\overline{G}_h = \overline{G}_h^*$, $N = 512$. In the last row $\overline{E}(N)$ gives the maximum over each column.

$\varepsilon \setminus N$	8	16	32	64	128	256
1·0	7·411E−04	1·843E−04	4·588E−05	1·133E−05	2·698E−06	5·395E−07
2^{-1}	7·305E−03	1·821E−03	4·538E−04	1·121E−04	2·669E−05	5·338E−06
2^{-2}	2·184E−02	5·459E−03	1·361E−03	3·362E−04	8·005E−05	1·601E−05
2^{-3}	3·086E−02	1·150E−02	3·064E−03	7·699E−04	1·844E−04	3·699E−05
2^{-4}	3·148E−02	3·433E−02	1·391E−02	3·630E−03	8·764E−04	1·760E−04
2^{-5}	3·149E−02	3·827E−02	3·325E−02	1·434E−02	3·597E−03	7·292E−04
2^{-6}–2^{-12}	3·149E−02	3·796E−02	3·275E−02	1·358E−02	4·015E−03	1·683E−03
$\overline{E}(N)$	3·149E−02	3·827E−02	3·325E−02	1·434E−02	4·015E−03	1·683E−03

6. Improved time accuracy

6.1 *A scheme based on defect correction*

In this section we construct a new discrete method based on defect correction, which also converges ε-uniformly to the solution of the boundary value problem, but with an order of accuracy (with respect to τ) higher than in (4.2).

The idea is similar to that in Hemker *et al.* (1997). For the difference scheme (3.2), (4.1) the error in the approximation of the partial derivative $(\partial/\partial t)\,u(x,t)$ is caused by the divided difference $\delta_{\bar t}\, z(x,t)$ and is associated with the truncation error given by the relation

$$\frac{\partial u}{\partial t}(x,t) - \delta_{\bar t}\, u(x,t) = 2^{-1}\,\tau \frac{\partial^2 u}{\partial t^2}(x,t) - 6^{-1}\,\tau^2 \frac{\partial^3 u}{\partial t^3}(x, t-\vartheta), \qquad (6.1)$$

where $\vartheta \in [0,\tau]$. Therefore, with the notation from Section 3, we now use for the approximation of $(\partial/\partial t)\,u(x,t)$ the expression

$$\delta_{\bar t}\, u(x,t) + \tau \delta_{\bar t \bar t}\, u(x,t)/2,$$

where $\delta_{\bar t \bar t}\, u(x,t) \equiv \delta_{t \bar t}\, u(x, t-\tau)$. Notice that $\delta_{t \bar t}\, u(x,t)$ is the second central divided difference. We can evaluate a better approximation than (3.2a) by defect correction

$$\Lambda_{(3.2)} z^c(x,t) = f(x,t) - 2^{-1} p(x,t)\tau \frac{\partial^2 u}{\partial t^2}(x,t), \qquad (6.2)$$

with $x \in \overline{\omega}$ and $t \in \overline{\omega}_0$, where $\overline{\omega}$ and $\overline{\omega}_0$ are as in (3.1); τ is the step-size of the mesh $\overline{\omega}_0$; $z^c(x,t)$ is the 'corrected' solution. Instead of $(\partial^2/\partial t^2)\,u(x,t)$ we shall use $\delta_{\bar t \bar t}\, z(x,t)$, where $z(x,t)$, $(x,t) \in G_{h(4.1)}$ is the solution of the difference scheme (3.2), (4.1). We may expect that the new solution $z^c(x,t)$ has an accuracy of $\mathcal{O}(\tau^2)$ with respect to the time variable. This is true, as will be shown in Section 6.3.

Moreover, in a similar way we can construct a difference approximation with a convergence order higher than two (with respect to the time variable) and $\mathcal{O}(N^{-2}\ln^2 N)$ with respect to the space variable ε-uniformly (see Section 6.2).

6.2 *Modified difference schemes of second-order accuracy in* τ

We denote by $\delta_{k\bar t} z(x,t)$ the backward difference of order k:

$$\delta_{0\bar t}\, z(x,t) = z(x,t),$$

$$\delta_{k\bar t}\, z(x,t) = \big(\delta_{k-1\,\bar t}\, z(x,t) - \delta_{k-1\,\bar t}\, z(x,t-\tau)\big)/\tau,$$

$$(x,t) \in \overline{G}_h, \qquad t \geqslant k\tau, \qquad k \geqslant 1.$$

When constructing difference schemes of second-order accuracy in τ in (6.2), instead of $(\partial^2/\partial t^2)u(x,t)$ we use $\delta_{2\bar t}\, z(x,t)$, which is the second divided difference of the solution to the discrete problem (3.2), (4.1). On the mesh $\overline{G}_h$ we consider the finite difference scheme (3.2), writing

$$\begin{aligned} \Lambda_{(3.2)} z^{(1)}(x,t) &= f(x,t), \qquad (x,t) \in G_h, \\ z^{(1)}(x,t) &= \varphi(x,t), \qquad (x,t) \in S_h. \end{aligned} \qquad (6.3)$$

Then for the boundary value problem (2.1) we now get for the difference equations for $t=\tau$ and $t \geqslant 2\tau$ respectively:

$$\Lambda_{(3.2)} z^{(2)}(x,t) = f(x,t) + \frac{p(x,t)}{2}\,\tau \frac{\partial^2 u}{\partial t^2}(x,0), \qquad t = \tau, \qquad (6.4)$$

$$\Lambda_{(3.2)}z^{(2)}(x,t) = f(x,t) + \frac{p(x,t)}{2}\,\tau\,\delta_{2\bar{t}}z^{(1)}(x,t), \qquad t \geqslant 2\tau, \qquad (x,t) \in G_h,$$

$$z^{(2)}(x,t) = \varphi(x,t), \qquad (x,t) \in S_h.$$

Here $z^{(1)}(x,t)$ is the solution of the discrete problem (6.3), (4.1), and the derivative $(\partial^2 u/\partial t^2)(x,0)$ is obtained from equation (2.1a). We shall call $z^{(2)}(x,t)$ the solution of difference scheme (6.4), (6.3), (4.1) (or in short, (6.4), (4.1)).

For simplicity, in the remainder of this section we take a homogeneous initial condition:

$$\varphi(x,0) = 0, \qquad x \in [0,1]. \tag{6.5}$$

Under the homogeneous initial condition (6.5), the following estimate holds for the solution of problem (6.4), (4.1)

$$\left| u(x,t) - z^{(2)}(x,t) \right| \leqslant M\left[N^{-2}\ln^2 N + \tau^2\right], \qquad (x,t) \in \overline{G}_h. \tag{6.6}$$

THEOREM 3 Let condition (6.5) hold and assume in equation (2.1) that $a \in H^{(\alpha+2n-1)}(\overline{G})$, c, p, $f \in H^{(\alpha+2n-2)}(\overline{G})$, $\varphi \in H^{(\alpha+2n)}(\overline{G})$, $\alpha > 4$, $n = 1$ and let condition (A.1) be satisfied for $n = 1$. Then for the solution of difference scheme (6.4), (4.1) the estimate (6.6) holds.

The proof of this theorem is found in Section A.2 of the Appendix.

6.3 *A difference scheme of third-order accuracy in time*

Analogously we construct a difference scheme with third-order accuracy in τ. On the mesh $\overline{G}_h$ we consider the difference scheme

$$\Lambda_{(3.2)}z^{(3)}(x,t) = f(x,t) + p(x,t)\left(C_{11}\tau\frac{\partial^2}{\partial t^2}u(x,0) + C_{12}\tau^2\frac{\partial^3 u}{\partial t^3}(x,0)\right), \qquad t = \tau,$$

(6.7a)

$$\Lambda_{(3.2)}z^{(3)}(x,t) = f(x,t) + p(x,t)\left(C_{21}\tau\frac{\partial^2 u}{\partial t^2}(x,0) + C_{22}\tau^2\frac{\partial^3 u}{\partial t^3}(x,0)\right), \qquad t = 2\tau,$$

$$\Lambda_{(3.2)}z^{(3)}(x,t) = f(x,t) + p(x,t)\left(C_{31}\tau\delta_{2\bar{t}}z^{(2)}(x,t) + C_{32}\tau^2\delta_{3\bar{t}}z^{(1)}(x,t)\right), \qquad t \geqslant 3\tau,$$

$$(x,t) \in G_h,$$

$$z^{(3)}(x,t) = \varphi(x,t), \qquad (x,t) \in S_h.$$

Here $z^{(1)}(x,t)$ and $z^{(2)}(x,t)$ are the solutions of problems (6.3), (4.1) and (6.4), (4.1) respectively, the derivatives $(\partial^2/\partial t^2)u(x,0)$, $(\partial^3/\partial t^3)u(x,0)$ are obtained from equation (2.1a), and the coefficients C_{ij} are determined below. They are chosen such that the following conditions are satisfied

$$\frac{\partial u}{\partial t}(x,t) = \delta_{\bar{t}}u(x,t) + C_{11}\tau\frac{\partial^2 u}{\partial t^2}(x,t-\tau) + C_{12}\tau^2\frac{\partial^3 u}{\partial t^3}(x,t-\tau) + \mathcal{O}(\tau^3),$$

$$\frac{\partial u}{\partial t}(x,t) = \delta_{\bar{t}}u(x,t) + C_{21}\tau\frac{\partial^2 u}{\partial t^2}(x,t-2\tau) + C_{22}\tau^2\frac{\partial^3 u}{\partial t^3}(x,t-2\tau) + \mathcal{O}(\tau^3),$$

$$\frac{\partial u}{\partial t}(x,t) = \delta_{\bar{t}}u(x,t) + C_{31}\tau\delta_{2\bar{t}}u(x,t) + C_{32}\tau^2\delta_{3\bar{t}}u(x,t) + \mathcal{O}(\tau^3).$$

It follows that

$$C_{11} = C_{21} = C_{31} = 1/2, \qquad C_{12} = C_{32} = 1/3, \qquad C_{22} = 5/6. \tag{6.7b}$$

We shall call $z^{(3)}(x,t)$ the solution of the difference scheme (6.7), (6.4), (6.3), (4.1) (or in short, (6.7), (4.1)).

Again we assume the homogeneous initial condition

$$\varphi(x,0) = 0, \qquad f(x,0) = 0, \qquad x \in [0,1]. \tag{6.8}$$

Under condition (6.8) the following estimate holds for the solution of difference scheme (6.7), (4.1)

$$\left|u(x,t) - z^{(3)}(x,t)\right| \leqslant M\left[N^{-2}\ln^2 N + \tau^3\right], \qquad (x,t) \in \overline{G}_h. \tag{6.9}$$

THEOREM 4 Let conditions (6.8) hold and assume in equation (2.1) that $a \in H^{(\alpha+2n-1)}(\overline{G})$, c, p, $f \in H^{(\alpha+2n-2)}(\overline{G})$, $\varphi \in H^{(\alpha+2n)}(\overline{G})$, $\alpha > 4$, $n = 2$ and let condition (A.1) be satisfied with $n = 2$. Then for the solution of scheme (6.7), (4.1) the estimate (6.9) is valid.

The proof of Theorem 4 is given in Section A.3 of the Appendix.

In a similar way we could construct difference schemes with an arbitrary high order of accuracy

$$\mathcal{O}(N^{-2}\ln^2 N + \tau^{n+1}), \qquad n > 2.$$

7. Numerical results for the time-accurate schemes

The solution of the problem in the half-strip,

$$L_{(5.1)}V(x,t) = 0, \qquad 0 < x < \infty, \qquad 0 < t \leqslant T, \tag{7.1}$$

$$V(0,t) = t^4, \qquad 0 < t \leqslant T, \qquad V(x,0) = 0, \qquad 0 \leqslant x < \infty,$$

is given by

$$V(x,t) = \operatorname{erfc}\left(\frac{x}{2\varepsilon\sqrt{t}}\right)\left(\frac{x^8}{1680\varepsilon^8} + \frac{x^6}{30\varepsilon^6}t + \frac{x^4}{2\varepsilon^4}t^2 + \frac{2x^2}{\varepsilon^2}t^3 + t^4\right) \tag{7.2}$$

$$-\frac{1}{\sqrt{\pi}}\exp\left(\frac{-x^2}{4\varepsilon^2 t}\right)\left(\frac{x^7}{840\varepsilon^7}t^{1/2} + \frac{9x^5}{140\varepsilon^5}t^{3/2} + \frac{37x^3}{42\varepsilon^3}t^{5/2} + \frac{93x}{35\varepsilon}t^{7/2}\right).$$

We consider the model problem

$$\begin{aligned} L_{(5.1)}u(x,t) &= 0, \quad (x,t) \in G, \\ u(x,t) &= V_{(7.2)}(x,t), \quad (x,t) \in S. \end{aligned} \tag{7.3}$$

Then the function $V_{(7.2)}(x,t)$ is the solution of problem (7.3).

The solution has a boundary layer character, and at the point $x = 1$, $V(x,t)$ is exponentially small in ε for $\varepsilon \to 0$.

According to Theorems 3 and 4, the difference schemes (6.4), (4.1) and (6.7), (4.1) converge respectively with order 2 and 3 with respect to τ. To demonstrate this effect numerically, we consider the schemes (6.3), (4.1); (6.4), (4.1) and (6.7), (4.1) for problem (7.3). We solve problem (5.1), using the schemes (6.3), (4.1); (6.4), (4.1) and (6.7), (4.1) for various values of N, K and ε.

As the solution of the boundary value problem (7.3) has a boundary layer on the left-hand side, for its solution we use the locally condensed mesh

$$\overline{G}_h^{(*)} = \overline{\omega}^{(*)} \times \overline{\omega}_0, \tag{7.4}$$

where $\overline{\omega}^{(*)} = \overline{\omega}^{(*)}(\sigma)$ is a special mesh, condensed in the neighbourhood of the left-hand side of the interval $[\,0, 1\,]$; σ is a parameter depending on ε and N. The mesh $\overline{\omega}^{(*)}(\sigma)$ is constructed as in Section 4, with the understanding that there is now only one boundary layer. We take $\sigma = \min[\,1/2,\ m\varepsilon \ln N\,]$, where m is an arbitrary positive number. Then for the solution z of the discrete problem (3.2), (7.4) we have the estimate:

$$|u(x,t) - z\,(x,t)| \leqslant M\left(N^{-2}\ln^2 N + \tau\right), \qquad (x,t) \in \overline{G}_h^{(*)}. \tag{7.5}$$

For the solution $z^{(1)}$ of the problem (6.3), (bneq.7.4) we have the following estimate:

$$\left|u(x,t) - z^{(1)}(x,t)\right| \leqslant M\left(N^{-2}\ln^2 N + \tau\right), \qquad (x,t) \in \overline{G}_h^{(*)}. \tag{7.6}$$

For the solution $z^{(2)}$ of the problem (6.4), (7.4), where $z^{(1)}(x,t)$ is the solution of problem (6.3), (7.4), the following estimate holds:

$$\left|u(x,t) - z^{(2)}(x,t)\right| \leqslant M\left(N^{-2}\ln^2 N + \tau^2\right), \qquad (x,t) \in \overline{G}_h^{(*)}. \tag{7.7}$$

For the solution $z^{(3)}$ of the discrete problem (6.7), (7.4), where $z^{(2)}(x,t)$ and $z^{(1)}(x,t)$ are the solutions of problems (6.4), (7.4) and (6.3), (7.4) respectively, the following estimate holds:

$$\left|u(x,t) - z^{(3)}(x,t)\right| \leqslant M\left(N^{-2}\ln^2 N + \tau^3\right), \qquad (x,t) \in \overline{G}_h^{(*)}. \tag{7.8}$$

The results from numerical experiments for the above model problem are given in Tables 2–5.

We know that the error consists of two contributions: one caused by the discretization of the time derivative and the other by the space derivative (put briefly, the time and space errors). From theory we know that the order of convergence is one for τ, and two for h. This dependence can be observed from the error tables, in regions where one component of error is negligible compared with the other. Thus, to see first-order convergence in τ, we should consider the errors where the contribution from the discretization of the space derivative is relatively small. Referring to Table 2, these errors are in the upper-right corner of the table.

TABLE 2

Table of errors $E(N, K, \varepsilon)$ *for scheme (6.3), (7.4).* $E(N, K, \varepsilon)$ *is defined by (5.2), where* $z(x,t) = z^{(1)}_{(6.3)}(x,t)$, $u^*(x,t) = V_{(7.2)}(x,t)$, $\overline{G}_h = \overline{G}^{(*)}_{h(7.4)}$.

		N								
ε	K	8	16	32	64	128	256	512	1024	2048
1	8	3·36-2	3·34-2	3·33-2	3·33-2	3·33-2	3·33-2	3·33-2	3·33-2	3·33-2
	16	1·79-2	1·75-2	1·74-2	1·74-2	1·74-2	1·74-2	1·74-2	1·74-2	1·74-2
	32	9·49-3	9·02-3	8·90-3	8·87-3	8·87-3	8·86-3	8·86-3	8·86-3	8·86-3
	64	5·17-3	4·65-3	4·52-3	4·49-3	4·48-3	4·48-3	4·48-3	4·48-3	4·48-3
	128	2·97-3	2·43-3	2·30-3	2·26-3	2·25-3	2·25-3	2·25-3	2·25-3	2·25-3
	256	1·86-3	1·31-3	1·17-3	1·14-3	1·13-3	1·13-3	1·13-3	1·13-3	1·13-3
	512	1·30-3	7·51-4	6·12-4	5·77-4	5·68-4	5·66-4	5·65-4	5·66-4	5·65-4
	1024	1·02-3	4·70-4	3·30-4	2·94-4	2·86-4	2·83-4	2·83-4	2·83-4	2·83-4
	2048	8·85-4	3·29-4	1·88-4	1·53-4	1·44-4	1·42-4	1·42-4	1·41-4	1·41-4
2^{-2}	8	5·76-2	4·87-2	4·65-2	4·61-2	4·60-2	4·59-2	4·59-2	4·59-2	4·59-2
	16	3·66-2	2·68-2	2·42-2	2·37-2	2·35-2	2·34-2	2·34-2	2·34-2	2·34-2
	32	2·58-2	1·55-2	1·27-2	1·21-2	1·19-2	1·18-2	1·18-2	1·18-2	1·18-2
	64	2·03-2	9·70-3	6·87-3	6·19-3	6·00-3	5·96-3	5·95-3	5·95-3	5·94-3
	128	1·76-2	6·81-3	3·94-3	3·22-3	3·04-3	2·99-3	2·98-3	2·98-3	2·98-3
	256	1·62-2	5·35-3	2·46-3	1·74-3	1·55-3	1·51-3	1·50-3	1·49-3	1·49-3
	512	1·55-2	4·62-3	1·73-3	9·94-4	8·08-4	7·62-4	7·50-4	7·47-4	7·47-4
	1024	1·51-2	4·26-3	1·36-3	6·21-4	4·35-4	3·89-4	3·77-4	3·74-4	3·74-4
	2048	1·50-2	4·07-3	1·17-3	4·35-4	2·49-4	2·02-4	1·91-4	1·88-4	1·87-4
2^{-4}	8	7·24-2	6·68-2	5·32-2	4·87-2	4·65-2	4·61-2	4·60-2	4·59-2	4·59-2
	16	5·66-2	4·67-2	3·26-2	2·68-2	2·42-2	2·37-2	2·35-2	2·34-2	2·34-2
	32	4·87-2	3·65-2	2·19-2	1·55-2	1·27-2	1·21-2	1·19-2	1·18-2	1·18-2
	64	4·48-2	3·14-2	1·65-2	9·70-3	6·87-3	6·19-3	6·00-3	5·96-3	5·95-3
	128	4·28-2	2·88-2	1·38-2	6·81-3	3·94-3	3·22-3	3·04-3	2·99-3	2·98-3
	256	4·19-2	2·75-2	1·24-2	5·35-3	2·46-3	1·74-3	1·55-3	1·51-3	1·50-3
	512	4·14-2	2·68-2	1·17-2	4·62-3	1·73-3	9·94-4	8·08-4	7·62-4	7·50-4
	1024	4·11-2	2·65-2	1·14-2	4·26-3	1·36-3	6·21-4	4·35-4	3·89-4	3·77-4
	2048	4·10-2	2·63-2	1·12-2	4·07-3	1·17-3	4·35-4	2·49-4	2·02-4	1·91-4
2^{-6}	8	7·24-2	6·68-2	5·32-2	4·92-2	4·70-2	4·62-2	4·60-2	4·60-2	4·59-2
&	16	5·66-2	4·67-2	3·26-2	2·72-2	2·47-2	2·38-2	2·35-2	2·35-2	2·34-2
2^{-8}	32	4·87-2	3·65-2	2·19-2	1·58-2	1·32-2	1·23-2	1·20-2	1·19-2	1·18-2
	64	4·48-2	3·14-2	1·65-2	1·00-2	7·33-3	6·40-3	6·09-3	5·99-3	5·96-3
	128	4·28-2	2·88-2	1·38-2	7·13-3	4·40-3	3·44-3	3·13-3	3·02-3	2·99-3
	256	4·19-2	2·75-2	1·24-2	5·68-3	2·92-3	1·96-3	1·64-3	1·54-3	1·51-3
	512	4·14-2	2·68-2	1·17-2	4·95-3	2·19-3	1·22-3	8·97-4	7·93-4	7·60-4
	1024	4·11-2	2·65-2	1·14-2	4·58-3	1·82-3	8·48-4	5·24-4	4·20-4	3·87-4
	2048	4·10-2	2·63-2	1·12-2	4·40-3	1·63-3	6·62-4	3·38-4	2·33-4	2·01-4

We say that the global error has the correct behaviour with respect to time if, by doubling K for fixed N, we obtain the ratio of the errors not less than some fixed number m_0. If e.g., for Table 2 we take $m_0 = 1{\cdot}7$, which is sufficiently close to two, the domains with correct and incorrect behaviour of the errors are separated by diagonals going in the direction from upper left to lower right.

From Table 2 we see that, for $\varepsilon = 1$, the domain with the correct behaviour of the error is almost the whole table, except for a few values in the lower-left corner. As ε decreases,

TABLE 3

Table of errors $E(N,K,\varepsilon)$ for scheme (6.4), (7.4). $E(N,K,\varepsilon)$ is defined by (5.2), where $z(x,t)=z^{(2)}_{(6.4)}(x,t)$, $u^*(x,t)=V_{(7.2)}(x,t)$, $\overline{G}_h=\overline{G}^{(*)}_{h(7.4)}$.

ε	K	N								
		8	16	32	64	128	256	512	1024	2048
1	8	6·64-3	6·18-3	6·04-3	6·01-3	6·00-3	6·00-3	6·00-3	6·00-3	6·00-3
	16	2·33-3	1·80-3	1·66-3	1·63-3	1·62-3	1·62-3	1·61-3	1·61-3	1·61-3
	32	1·16-3	6·06-4	4·64-4	4·30-4	4·21-4	4·19-4	4·19-4	4·18-4	4·18-4
	64	8·50-4	2·94-4	1·53-4	1·18-4	1·09-4	1·07-4	1·07-4	1·06-4	1·06-4
	128	7·71-4	2·15-4	7·38-5	3·86-5	2·97-5	2·75-5	2·70-5	2·68-5	2·68-5
	256	7·52-4	1·95-4	5·38-5	1·85-5	9·64-6	7·43-6	6·88-6	6·74-6	6·70-6
	512	7·47-4	1·90-4	4·87-5	1·34-5	4·58-6	2·38-6	1·82-6	1·69-6	1·65-6
	1024	7·45-4	1·88-4	4·75-5	1·22-5	3·32-6	1·11-6	5·65-7	4·33-7	4·01-7
	2048	7·45-4	1·88-4	4·72-5	1·18-5	3·00-6	7·93-7	2·67-7	1·56-7	1·35-7
2^{-2}	8	2·13-2	1·08-2	7·98-3	7·27-3	7·09-3	7·05-3	7·04-3	7·04-3	7·04-3
	16	1·65-2	5·72-3	2·84-3	2·11-3	1·92-3	1·88-3	1·87-3	1·86-3	1·86-3
	32	1·52-2	4·36-3	1·46-3	7·26-4	5·41-4	4·95-4	4·83-4	4·80-4	4·80-4
	64	1·49-2	4·01-3	1·11-3	3·70-4	1·84-4	1·37-4	1·25-4	1·23-4	1·22-4
	128	1·48-2	3·92-3	1·02-3	2·79-4	9·28-5	4·62-5	3·45-5	3·16-5	3·09-5
	256	1·48-2	3·90-3	9·94-4	2·50-4	6·99-5	2·32-5	1·16-6	8·66-6	7·93-6
	512	1·48-2	3·89-3	9·88-4	2·51-4	6·41-5	1·75-5	5·83-6	2·92-6	2·19-6
	1024	1·48-2	3·89-3	9·87-4	2·49-4	6·27-5	1·61-5	4·40-6	1·49-6	7·80-7
	2048	1·48-2	3·89-3	9·86-4	2·49-4	6·23-5	1·57-5	4·04-6	1·15-6	4·50-7
2^{-4}	8	4·49-2	3·21-2	1·76-2	1·08-2	7·98-3	7·27-3	7·09-3	7·05-3	7·04-3
	16	4·19-2	2·77-2	1·28-2	5·72-3	2·84-3	2·19-3	1·92-3	1·88-3	1·87-3
	32	4·12-2	2·66-2	1·15-2	4·36-3	1·46-3	7·26-4	5·41-4	4·95-4	4·83-4
	64	4·10-2	2·63-2	1·11-2	4·01-3	1·11-3	3·70-4	1·84-4	1·37-4	1·25-4
	128	4·09-2	2·62-2	1·10-2	3·92-3	1·02-3	2·79-4	9·28-5	4·62-5	3·45-5
	256	4·09-2	2·62-2	1·10-2	3·90-3	9·94-4	2·56-4	6·99-5	2·32-5	1·16-5
	512	4·09-2	2·62-2	1·10-2	3·89-3	9·88-4	2·51-4	6·41-5	1·75-5	5·83-6
	1024	4·09-2	2·62-2	1·10-2	3·89-3	9·87-4	2·49-4	6·27-5	1·61-5	4·40-6
	2048	4·09-2	2·62-2	1·10-2	3·89-3	9·86-4	2·49-4	6·23-5	1·57-5	4·04-6
2^{-6}	8	4·49-2	3·21-2	1·76-2	1·11-3	8·34-3	7·50-3	7·18-3	7·08-3	7·05-3
&	16	4·19-2	2·77-2	1·28-2	6·04-3	3·27-3	2·34-3	2·01-3	1·91-3	1·88-3
2^{-8}	32	4·12-2	2·66-2	1·15-2	4·69-3	1·92-3	9·55-4	6·30-4	5·26-4	4·93-4
	64	4·10-2	2·63-2	1·11-2	4·33-3	1·57-3	5·98-4	2·73-4	1·68-4	1·36-4
	128	4·09-2	2·62-2	1·10-2	4·25-3	1·48-3	5·07-4	1·82-4	7·73-5	4·47-5
	256	4·09-2	2·62-2	1·10-2	4·22-3	1·46-3	4·84-4	1·59-4	5·44-5	2·18-5
	512	4·09-2	2·62-2	1·10-2	4·22-3	1·45-3	4·78-4	1·53-4	4·86-5	1·61-5
	1024	4·09-2	2·62-2	1·10-2	4·22-3	1·45-3	4·77-4	1·52-4	4·72-5	1·46-5
	2048	4·09-2	2·62-2	1·10-2	4·22-3	1·45-3	4·77-4	1·51-4	4·68-5	1·43-5

the domain of correct behaviour of the error tends to decrease. This can be explained by the relative increasing influence of the space error for smaller ε. In the case of $\varepsilon \leqslant 2^{-6}$ the domain of correct behaviour of the error no longer changes. Thus, we can observe ε-uniform convergence, of order (approximately) one with respect to time.

Now we consider Table 3, which gives the errors for $z^{(2)}(x,t)$, that is the corrected solution. Note that, in principle, the time correction does not improve the accuracy with respect to the space variable. By the correction we improve only the part of the error that is caused by the approximation of the time derivative, depending on τ, and we observe the improvement only when the space-dependent contribution of the error is small in comparison with the time-dependent part. For $\varepsilon = 1$ the space error is relatively small, and the improvement in accuracy can be seen immediately.

Analysing Table 3, we define the domain where the error behaviour is correct, e.g., by the value $m_0 = 3$ (see above). This number is larger than 2 and smaller than 4, corresponding to the second-order convergence in τ. When the parameter ε decreases, the domain with correct behaviour of the error decreases. Note that, for small ε, the part of Table 3 with the correct behaviour of the error is much reduced because the effect of the space error is relatively large for the corrected solution. Again, for $\varepsilon \leqslant 2^{-6}$ the errors for the same N, K do not change. Consequently, we see the ε-uniform effect of the improvement of accuracy, and the order of convergence with respect to the time-step size is about two.

In Table 4 we recognize the third-order time-accuracy described in Section 6.3. We define the domain of correct behaviour of the error by $m_0 = 5$ (for third-order convergence we should take $4 < m_0 < 8$). For $\varepsilon = 1$ we see that the errors for the same N, K in Table 4 are smaller than those in Tables 2 and 3. (In the same way that the errors in Table 3 are smaller than those in Table 2.)

Therefore, the time error in Table 4 is much smaller than in Tables 2 and 3. With decreasing ε the domain of correct behaviour of the error decreases and, for given N and K, practically disappears if $\varepsilon \leqslant 2^{-6}$. Note that the errors in Table 4 for $\varepsilon \leqslant 2^{-6}$ are close to the errors in Table 3 in that part of the table where the errors almost do not vary with doubling K. In the remainder the errors in Table 4 are smaller than the corresponding errors in Table 3.

Comparing Tables 2, 3 and 4, we see that: (i) for all ε and N, by $K = 16$ the errors for $z^{(3)}(x, t)$ are smaller than those for $z^{(1)}(x, t)$ at $K = 2048$; (ii) the order of ε-uniform convergence with respect to τ is higher for scheme (6.4), (7.4) than that for scheme (6.3), (7.4), and the order for scheme (6.7), (7.4) is higher than for scheme (6.4), (7.4); (iii) the order of convergence with respect to τ increases for the functions $z^{(k)}(x, t)$ with increasing k; (iv) for sufficiently large N the order of convergence with respect to the space variable is nearly two, uniformly in ε.

Because the space error is smaller for $\varepsilon = 1$, this case shows more clearly the effect of the defect correction in time. Hence, for illustration, Table 5 is derived from the values $E(N_i, K_j, \varepsilon)$ shown in Tables 2, 3 and 4, for $\varepsilon = 1$. In Table 5 we give the ratios

$$r = E(N_i, K_j, 1)/E(N_i, K_{j+1}, 1) > m_0,$$

where the value m_0 is chosen as above. From Table 5 we see that the function $E(N, K, \varepsilon)$ for $N = 2048$ for $z^{(1)}$ decreases by a factor 2, when K is doubled, for $z^{(2)}$ it decreases by a factor 4, and for $z^{(3)}$ by a factor of 8 for not too large K.

When the parameter ε increases, a similar behaviour of the function $E(N, K, \varepsilon)$ is observed, however for much larger values of the number N. From the tables we can see that the order of convergence with respect to the space variable is close to 2, uniformly in ε, for sufficiently large N.

Comparing the present results with those in Hemker *et al.* (1997) we can make the following remarks: (i) for the model problem (7.3), the errors for the schemes (6.3), (7.4);

TABLE 4

Table of errors $E(N, K, \varepsilon)$ *for scheme (6.7), (7.4).* $E(N, K, \varepsilon)$ *is defined by (5.2), where* $z(x, t) = z^{(3)}_{(6.7)}(x, t)$, $u^*(x, t) = V_{(7.2)}(x, t)$, $\overline{G}_h = \overline{G}^{(*)}_{h(7.4)}$.

		N								
ε	K	8	16	32	64	128	256	512	1024	2048
1	8	1·64-3	1·11-3	9·66-4	9·32-4	9·23-4	9·21-4	9·20-4	9·20-4	9·20-4
	16	8·61-4	3·08-4	1·67-4	1·32-4	1·23-4	1·21-4	1·21-4	1·21-4	1·20-4
	32	7·60-4	2·03-4	6·23-5	1·70-5	1·83-5	1·61-5	1·55-5	1·54-5	1·53-5
	64	7·47-4	1·90-4	4·90-5	1·37-5	4·83-6	2·63-6	2·08-6	1·95-6	1·91-6
	128	7·45-4	1·88-4	4·73-5	1·20-5	3·14-6	9·31-7	3·99-7	2·78-7	2·51-7
	256	7·45-4	1·88-4	4·71-5	1·18-5	2·92-6	7·18-7	2·04-7	1·16-7	1·01-7
	512	7·45-4	1·88-4	4·71-5	1·17-5	2·90-6	6·92-7	1·81-7	1·06-7	9·24-8
	1024	7·45-4	1·88-4	4·71-5	1·17-5	2·89-6	6·88-7	1·78-7	1·05-7	9·13-8
	2048	7·45-4	1·88-4	4·71-5	1·17-5	2·89-6	6·88-7	1·78-7	1·05-7	9·11-8
2^{-2}	8	1·59-2	5·01-3	2·13-3	1·40-3	1·21-3	1·17-3	1·16-3	1·15-3	1·15-3
	16	1·49-2	4·04-3	1·13-3	3·96-4	2·10-4	1·63-4	1·52-4	1·49-4	1·48-4
	32	1·48-2	3·91-3	1·00-3	2·67-4	8·10-5	3·44-5	2·28-5	1·99-5	1·92-5
	64	1·48-2	3·89-3	9·88-4	2·51-4	6·46-5	1·80-5	6·31-6	3·41-6	2·69-6
	128	1·48-2	3·89-3	9·86-4	2·49-4	6·25-5	1·59-5	4·22-6	1·33-6	6·31-7
	256	1·48-2	3·89-3	9·86-4	2·49-4	6·23-5	1·56-5	3·96-6	1·07-6	6·31-7
	512	1·48-2	3·89-3	9·86-4	2·49-4	6·22-5	1·56-5	3·93-6	1·04-6	3·49-7
	1024	1·48-2	3·89-3	9·86-4	2·49-4	6·22-5	1·56-5	3·92-6	1·03-6	3·45-7
	2048	1·48-2	3·89-3	9·86-4	2·49-4	6·22-5	1·56-5	3·92-6	1·03-6	3·45-7
2^{-4}	8	4·16-2	2·72-2	1·21-2	5·01-3	2·13-3	1·40-3	1·21-3	1·17-3	1·16-3
	16	4·10-2	2·63-2	1·12-2	4·04-3	1·13-3	3·96-4	2·10-4	1·63-4	1·52-4
	32	4·09-2	2·62-2	1·10-2	3·91-3	1·00-3	2·67-4	8·10-5	3·44-5	2·28-5
	64	4·09-2	2·62-2	1·10-2	3·89-3	9·88-4	2·51-4	6·46-5	1·80-5	6·31-6
	128	4·09-2	2·62-2	1·10-2	3·89-3	9·86-4	2·49-4	6·25-5	1·59-5	4·22-6
	256	4·09-2	2·62-2	1·10-2	3·89-3	9·86-4	2·49-4	6·23-5	1·56-5	3·96-6
	512	4·09-2	2·62-2	1·10-2	3·89-3	9·86-4	2·49-4	6·22-5	1·56-5	3·93-6
	1024	4·09-2	2·62-2	1·10-2	3·89-3	9·86-4	2·49-4	6·22-5	1·56-5	3·92-6
	2048	4·09-2	2·62-2	1·10-2	3·89-3	9·86-4	2·49-4	6·22-5	1·56-5	3·92-6
2^{-6}	8	4·16-2	2·72-2	1·21-2	5·34-3	2·59-3	1·63-3	1·30-3	1·20-3	1·17-3
&	16	4·10-2	2·63-2	1·12-2	4·36-3	1·60-3	6·22-4	2·99-4	1·94-4	1·62-4
2^{-8}	32	4·09-2	2·62-2	1·10-2	4·23-3	1·47-3	4·95-4	1·70-4	6·55-5	3·30-5
	64	4·09-2	2·62-2	1·10-2	4·22-3	1·45-3	4·79-4	1·54-4	4·91-5	1·65-5
	128	4·09-2	2·62-2	1·10-2	4·22-3	1·45-3	4·77-4	1·51-4	4·70-5	1·45-5
	256	4·09-2	2·62-2	1·10-2	4·22-3	1·45-3	4·76-4	1·51-4	4·68-5	1·42-5
	512	4·09-2	2·62-2	1·10-2	4·22-3	1·45-3	4·76-4	1·51-4	4·67-5	1·42-5
	1024	4·09-2	2·62-2	1·10-2	4·22-3	1·45-3	4·76-4	1·51-4	4·67-5	1·42-5
	2048	4·09-2	2·62-2	1·10-2	4·22-3	1·45-3	4·76-4	1·51-4	4·67-5	1·42-5

(6.4), (7.4) and (6.7), (7.4) are comparable with those in Hemker *et al.* (1997), but now we can show results for a wider range of parameters N and K; (ii) in Hemker *et al.* (1997) defect correction schemes were considered for sequences of *embedded time-refined meshes* (meshes that were refined in the time variable). Here we use a *single time mesh*, both for the corrected solution and for the auxiliary solutions. This essentially simplifies the structure of the schemes and consequently their computer implementation. Finally: (iii) in Hemker *et al.* (1997) an order of convergence with respect to the space variable of $\mathcal{O}(N^{-1} \ln N)$

TABLE 5

Table of the ratios $r = E(N_i, K_j, 1)/E(N_i, K_{j+1}, 1) > m_0$. Here $E(N_i, K_j, 1)$ is the error in $z^{(k)}(x,t)$, $k = 1, 2, 3$, as in Tables 2–4; m_0 is as described in the text. Where the space error dominates, $r < m_0$ is indicated by $*$.

		N								
k	K	8	16	32	64	128	256	512	1024	2048
1	8	1·88	1·91	1·92	1·92	1·92	1·92	1·92	1·92	1·92
	16	1·88	1·94	1·95	1·96	1·96	1·96	1·96	1·96	1·96
	32	1·84	1·94	1·97	1·98	1·98	1·98	1·98	1·98	1·98
	64	1·74	1·91	1·97	1·98	1·99	1·99	1·99	1·99	1·99
	128	*	1·85	1·95	1·98	1·99	1·99	1·99	1·99	1·99
	256	*	1·75	1·92	1·98	1·99	2·00	2·00	2·00	2·00
	512	*	*	1·86	1·96	1·99	2·00	2·00	2·00	2·00
	1024	*	*	1·75	1·92	1·98	1·99	2·00	2·00	2·00
2	8	*	3·43	3·64	3·70	3·71	3·72	3·72	3·72	3·72
	16	*	*	3·57	3·78	3·84	3·85	3·86	3·86	3·86
	32	*	*	3·03	3·64	3·85	3·91	3·93	3·93	3·93
	64	*	*	*	3·06	3·68	3·89	3·95	3·97	3·97
	128	*	*	*	*	3·09	3·71	3·92	3·98	4·00
	256	*	*	*	*	*	3·13	3·77	3·99	4·06
	512	*	*	*	*	*	*	3·23	3·89	4·12
	1024	*	*	*	*	*	*	*	*	*
3	8	*	*	5·78	7·05	7·48	7·60	7·63	7·64	7·64
	16	*	*	*	*	6·76	7·55	7·78	7·84	7·85
	32	*	*	*	*	*	6·11	7·45	7·90	8·02
	64	*	*	*	*	*	*	5·22	7·00	7·63
	128	*	*	*	*	*	*	*	*	*
	256	*	*	*	*	*	*	*	*	*
	512	*	*	*	*	*	*	*	*	*
	1024	*	*	*	*	*	*	*	*	*

could be shown theoretically. Here, with the simpler defect correction schemes, a better theoretical order of convergence for the space variable, $\mathcal{O}(N^{-2} \ln^2 N)$, is achieved.

8. Conclusions

In this paper we showed a possible defect correction procedure that can easily be implemented in order to improve the time accuracy, whilst still retaining ε-uniform second-order accuracy in the space discretization, for a parabolic PDE.

The approximation error consists of two components. One is due to the discretization of the space variable and the other is due to the time discretization. The defect correction process only improves the accuracy with respect to the time and does not change the approximation with respect to the space variable. Therefore, by application of the defect correction, the principal part of the total error becomes that due to the approximation of the space variable.

By using the defect correction we are able to increase the accuracy of the approximate solution essentially, i.e. from 1st to 2nd and 3rd order in τ. In the present paper we use the same time mesh for the corrected solution and for the auxiliary solutions. Therefore the structure of the present schemes is much simpler than that of those introduced in Hemker *et al.* (1997). Numerical results illustrate that, also in practice, the order of convergence with respect to the space variable is close to two.

REFERENCES

DOOLAN, E. P., MILLER, J. J. H., & SCHILDERS, W. H. A. 1980 *Uniform Numerical Methods for Problems with Initial and Boundary Layers*. Dublin: Boole.

FARRELL, P. A., HEMKER, P. W., & SHISHKIN, G. I. 1996a Discrete approximations for singularly perturbed boundary value problems with parabolic layers, I. *J. Comput. Math.* **14**, 71–97.

FARRELL, P. A., HEMKER, P. W., & SHISHKIN, G. I. 1996b Discrete approximations for singularly perturbed boundary value problems with parabolic layers, II. *J. Comput. Math.* **14**, 183–194.

FARRELL, P. A., HEMKER, P. W., & SHISHKIN, G. I. 1996c Discrete approximations for singularly perturbed boundary value problems with parabolic layers, III. *J. Comput. Math.* **14**, 273–290.

FARRELL, P. A., MILLER, J. J. H., O'RIORDAN, E., & SHISHKIN, G. I. 1996d A uniformly convergent finite difference scheme for a singularly perturbed semilinear equation. *SIAM J. Numer. Anal.* **33**, 1135–1149.

HEMKER, P. W. & SHISHKIN, G. I. 1994 On a class of singularly perturbed boundary value problems for which an adaptive mesh technique is necessary. *Proc. Second Int. Colloquium on Numerical Analysis* (D. Bainov and V. Covachev, eds). VSP, International Science Publishers, pp 83–92.

HEMKER, P. W., SHISHKIN, G. I., & SHISHKINA, L. P. 1997 The use of defect correction for the solution of parabolic singular perturbation problems. *Z. Angewandte Mathematik Mechanik* **76**, 59–74.

LADYZHENSKAYA, O. A. & URAL'TSEVA, N. N. 1973 *Linear and Quasi-Linear Equations of Elliptic Type*. Moscow: Nauka (in Russian).

MILLER, J. J. H., O'RIORDAN, E., & SHISHKIN, G. I. 1996 *Numerical Methods For Singular Perturbation Problems: Error Estimates in the Maximum Norm for Linear Problems in One and Two Dimensions*. Singapore: World Scientific,.

MORTON, K. W. 1996 *Numerical Solution of Convection–Diffusion Problems*, vol. 12 of *Applied Mathematics and Mathematical Computation*. London: Chapman and Hall.

ROOS, H.-G., STYNES, M., & TOBISKA, L. 1996 *Numerical Methods for Singularly Perturbed Diffferential Equations: Convection–Diffusion and Flow Problems* (*Springer Series in Computational Mathematics 24*). Berlin: Springer.

SAMARSKI, A. A. 1989 *Theory of Difference Scheme*. Moscow: Nauka (in Russian).

SHISHKIN, G. I. 1983 Difference scheme on a non-uniform grid for the differential equation with a small parameter at the highest derivative. *Zh. Vychisl. Mat. Mat. Fiz.* **23**, 609–619 (in Russian).

SHISHKIN, G. I. 1987 Approximation of solutions to singularly perturbed boundary value problems with corner boundary layers. *Zh. Vychisl. Mat. Mat. Fiz.* **27**, 1360–1374 (in Russian).

SHISHKIN, G. I. 1952 Mesh Approximation of Singularly Perturbed Elliptic and Parabolic Equations. Ekaterinburg: Ur. O. RAN (in Russian).

Appendix

A.1 *Estimates of the solution and its derivatives*

In this appendix we rely on the *a priori* estimates for the solution of problem (2.1) on the domain $G = D \times [0, T]$, and its derivatives as derived for elliptic and parabolic equations in Shishkin (1987, 1992).

We denote by $H^{(\alpha)}(\overline{G}) = H^{\alpha,\alpha/2}(\overline{G})$ the Hölder space, where α is an arbitrary positive number (Ladyzhenskaya & Ural'tseva, 1973). We suppose that the functions $f(x,t)$ and $\varphi(x,t)$ satisfy compatibility conditions at the corner points, so that the solution of the boundary value problem is smooth for every fixed value of the parameter ε.

For simplicity, we assume that at the corner points $S_0 \cap \overline{S}_1$ the following conditions hold

$$\frac{\partial^k}{\partial x^k}\varphi(x,t) = \frac{\partial^{k_0}}{\partial t^{k_0}}\varphi(x,t) = 0, \qquad k + 2k_0 \leqslant [\alpha] + 2n, \tag{A.1}$$

$$\frac{\partial^{k+k_0}}{\partial x^k\,\partial t^{k_0}} f(x,t) = 0, \qquad k + 2k_0 \leqslant [\alpha] + 2n - 2,$$

where $[\alpha]$ is the integer part of a number α, $\alpha > 0$, $n \geqslant 0$ is an integer. We also suppose that $[\alpha] + 2n \geqslant 2$.

Using interior *a priori* estimates and estimates up to the boundary for the regular function $\widetilde{u}(\xi,t)$ (cf. Ladyzhenskaya & Ural'tseva, 1973), where $\widetilde{u}(\xi,t) = u(x(\xi),t)$, $\xi = x/\varepsilon$, we find for $(x,t) \in \overline{G}$ the estimate

$$\left|\frac{\partial^{k+k_0}}{\partial x^k\,\partial t^{k_0}} u(x,t)\right| \leqslant M\varepsilon^{-k}, \qquad k + 2k_0 \leqslant 2n + 4, \qquad n \geqslant 0. \tag{A.2}$$

This estimate holds, for example, for

$$u \in H^{(2n+4+\nu)}(\overline{G}), \qquad \nu > 0, \tag{A.3}$$

where ν is some small number.

For example, (A.3) is guaranteed for the solution of (2.1) if the coefficients satisfy $a \in H^{(\alpha+2n-1)}(\overline{G})$, $c, p, f \in H^{(\alpha+2n-2)}(\overline{G})$, $\varphi \in H^{(\alpha+2n)}(\overline{G})$, $\alpha > 4$, $n \geqslant 0$ and condition (A.1) is fulfilled.

In fact we need a more accurate estimate than (A.2). Therefore, we represent the solution of the boundary value problem (2.1) in the form of the sum

$$u(x,t) = U(x,t) + W(x,t), \qquad (x,t) \in \overline{G}, \tag{A.4}$$

where $U(x,t)$ represents the regular part, and $W(x,t)$ the singular part, i.e. the parabolic boundary layer. The function $U(x,t)$ is the smooth solution of equation (2.1a) satisfying condition (2.1b) for $t = 0$. For example, under suitable assumptions for the data of the problem, we can consider the solution of the Dirichlet boundary value problem for equation (2.1a) smoothly extended to the domain $\overline{G}^*$ (where $\overline{G}^*$ is a sufficiently large neighbourhood of $\overline{G}$). On the domain $\overline{G}$ the coefficients and the initial value of the extended problem are the same as for (2.1). Then the function $U(x,t)$ is the restriction (on $\overline{G}$) of the solution to the extended problem, and $U \in H^{(2n+4+\nu)}(\overline{G})$, $\nu > 0$. The function $W(x,t)$ is the solution of a boundary value problem for the parabolic equation

$$L_{(2.1)} W(x,t) = 0, \qquad (x,t) \in G, \tag{A.5}$$

$$W(x,t) = u(x,t) - U(x,t), \qquad (x,t) \in S.$$

If (A.1) is true then $W \in H^{(2n+4+\nu)}(\overline{G})$. Now, for the functions $U(x,t)$ and $W(x,t)$ we derive the estimates

$$\left| \frac{\partial^{k+k_0}}{\partial x^k \, \partial t^{k_0}} U(x,t) \right| \leqslant M, \tag{A.6}$$

and

$$\left| \frac{\partial^{k+k_0}}{\partial x^k \, \partial t^{k_0}} W(x,t) \right| \leqslant M \varepsilon^{-k} \exp(-m_{(A.7)} \varepsilon^{-1} r(x,\gamma)), \tag{A.7}$$

$$(x,t) \in \overline{G}, \qquad k + 2k_0 \leqslant 2n + 4,$$

where $r(x,\gamma)$ is the distance between the point $x \in [0,1]$ and the set γ which represents the endpoints of the segment $[0,1]$, $m_{(A.7)}$ is a sufficiently small, positive number. The estimates (A.6) and (A.7) hold, for example, when

$$U,\ W \in H^{(2n+4+\nu)}(\overline{G}), \qquad \nu > 0. \tag{A.8}$$

The inclusions (A.8) are guaranteed if $a \in H^{(\alpha+2n-1)}(\overline{G})$, $c,\ p,\ f \in H^{(\alpha+2n-2)}(\overline{G})$, $\varphi \in H^{(\alpha+2n)}(\overline{G})$, $\alpha > 4$, $n \geqslant 0$ and condition (A.1) is fulfilled. We summarize these results in the following theorem.

THEOREM 5 Assume in equation (2.1) that $a \in H^{(\alpha+2n-1)}(\overline{G})$, $c,\ p,\ f \in H^{(\alpha+2n-2)}(\overline{G})$, $\varphi \in H^{(\alpha+2n)}(\overline{G})$, $\alpha > 4$, $n \geqslant 0$ and let condition (A.1) be fulfilled. Then, for the solution, $u(x,t)$, of problem (2.1), and for its components in the representation (A.4), it follows that $u,\ U,\ W \in H^{(\alpha+2n)}(\overline{G})$ and that the estimates (A.2), (A.6), (A.7) hold.

The proof of the theorem is similar to the proof in Shishkin (1992), where the equation

$$\varepsilon^2 a(x,t) \frac{\partial^2 u}{\partial x^2}(x,t) - c(x,t) u(x,t) - p(x,t) \frac{\partial u}{\partial t}(x,t) = f(x,t)$$

was considered.

A.2 *The proof of Theorem 3*

Let us show that the function $\delta_{\bar{t}} z(x,t)$, where $z(x,t) = z_{(6.3)}(x,t)$ is the solution of the difference problem (6.3), approximates the function $\delta_{\bar{t}} u(x,t)$ ε-uniformly. For simplicity we assume $a(x,t)$ to be constant on $\overline{G}$. The function $\delta_{\bar{t}} z(x,t)$ is the solution of the difference problem

$$\Lambda_{(A.9)} \delta_{\bar{t}} z(x,t) = f_{(A.9)}(x,t), \qquad (x,t) \in G_h^{[1]}, \tag{A.9a}$$

$$\delta_{\bar{t}} z(x,t) = \varphi_{(A.9)}(x,t), \qquad (x,t) \in S_h^{[1]}. \tag{A.9b}$$

Here

$$\overline{G}_h^{[k]} = \overline{G}_h \cap \{t \geqslant k\tau\}, \qquad G_h^{[k]} = G_h \cap \{t > k\tau\}, \qquad S_h^{[k]} = \overline{G}_h^{[k]} \setminus G_h^{[k]}, \qquad k \geqslant 1,$$

$$\Lambda_{(A.9)}\delta_{\overline{t}} z(x,t) \equiv \left\{ \varepsilon^2 a\, \delta_{\overline{x}\hat{x}} - \check{c}(x,t) - p_{\overline{t}}(x,t) - \check{p}(x,t)\,\delta_{\overline{t}} \right\} \delta_{\overline{t}} z(x,t),$$

$$f_{(A.9)}(x,t) = f_{\overline{t}}(x,t) + c_{\overline{t}}(x,t) z(x,t),$$

$$\varphi_{(A.9)}(x,t) = \varphi_{\overline{t}}(x,t), \qquad x = 0,\, d, \qquad (x,t) \in S_h^{[1]},$$

$$\varphi_{(A.9)}(x,t) = \varphi^0_{(A.9)}(x) \equiv \tau^{-1}\left[z(x,\tau) - \varphi(x,0) \right], \qquad t = \tau, \qquad (x,t) \in S_h^{[1]},$$

$$\check{v}(x,t) = v(x, t - \tau).$$

The function $\delta_{\overline{t}} u(x,t) \equiv [u(x,t) - u(x,t-\tau)]/\tau,\ (x,t) \in \overline{G},\ t \geqslant \tau$ is the solution of the differential problem

$$L_{(A.10)}\delta_{\overline{t}} u(x,t) = f_{(A.10)}(x,t), \qquad (x,t) \in G^{[1]}, \tag{A.10a}$$

$$\delta_{\overline{t}} u(x,t) = \varphi_{(A.10)}(x,t), \qquad (x,t) \in S^{[1]}. \tag{A.10b}$$

Here

$$\overline{G}^{[k]} = \overline{G} \cap \{t \geqslant k\tau\}, \qquad G^{[k]} = G \cap \{t > k\tau\}, \qquad S^{[k]} = \overline{G}^{[k]} \setminus G^{[k]}, \qquad k \geqslant 1,$$

$$L_{(A.10)}\delta_{\overline{t}} u(x,t) \equiv \varepsilon^2 a \frac{\partial^2}{\partial x^2} - \check{c}(x,t) - p_{\overline{t}}(x,t) - \check{p}(x,t)\frac{\partial}{\partial t}\delta_{\overline{t}} u(x,t),$$

$$f_{(A.10)}(x,t) = f_{\overline{t}}(x,t) + c_{\overline{t}}(x,t) u(x,t) + p_{\overline{t}}(x,t)\left(\frac{\partial u}{\partial t}(x,t) - \delta_{\overline{t}} u(x,t) \right),$$

$$\varphi_{(A.10)}(x,t) = \varphi_{\overline{t}}(x,t), \qquad x = 0,\, d, \qquad (x,t) \in S^{[1]},$$

$$\varphi_{(A.10)}(x,t) = \varphi^0_{(A.10)}(x) \equiv \tau^{-1}\left[u(x,\tau) - \varphi(x,0) \right], \qquad t = \tau, \qquad (x,t) \in S^{[1]}.$$

Let us estimate

$$\varphi^0_{(A.10)}(x) - \varphi^0_{(A.9)}(x) = \tau^{-1}\omega(x,\tau),$$

where

$$\omega(x,t) = u(x,t) - z(x,t), \qquad (x,t) \in \overline{G}_h.$$

The function $\omega(x,t)$ is the solution of the problem

$$\Lambda_{(6.3)}\,\omega(x,t) = (\Lambda_{(6.3)} - L_{(2.1)})\, u(x,t), \qquad (x,t) \in G_h,$$

$$\omega(x,t) = 0, \qquad (x,t) \in S_h.$$

The above assumptions and Theorem 5 lead to the estimate of the truncation error

$$\left| \left(\Lambda_{(6.3)} - L_{(2.1)} \right) u(x,t) \right| \leqslant M\left[N^{-1}\ln N + \tau \right], \qquad (x,t) \in G_h.$$

Using the maximum principle we estimate $\omega(x,t)$

$$|\omega(x,t)| \leqslant M\left[N^{-1}\ln N + \tau\right]t, \qquad (x,t) \in \overline{G}_h.$$

Further, for the derivatives we proceed similarly. On the boundary we have

$$\left|\delta_{\bar{t}}u(x,\tau) - \delta_{\bar{t}}z(x,\tau)\right| = \left|\varphi^0_{(A.10)}(x) - \varphi^0_{(A.9)}(x)\right| \leqslant M\left[N^{-1}\ln N + \tau\right],$$

$$(x,t) \in S_h^{[1]}, \qquad t = \tau,$$

i.e. the function $\delta_{\bar{t}}z(x,\tau)$ approximates $\delta_{\bar{t}}u(x,\tau)$ ε-uniformly. Now, it is easy to see that the difference problem (A.9) approximates the solution of the differential equation for the divided difference (A.10). Thus, using the same argument as above, we derive the estimate

$$\left|\delta_{\bar{t}}u(x,t) - \delta_{\bar{t}}z(x,t)\right| \leqslant M\left[N^{-1}\ln N + \tau\right], \qquad (x,t) \in \overline{G}_h^{[1]}.$$

Now, for the 2nd difference derivative we show that under condition (6.5) the function $\delta_{2\bar{t}}\,z(x,t)$ approximates the function $\delta_{2\bar{t}}\,u(x,t)$ ε-uniformly on the set $\overline{G}_h^{[2]}$. So, the functions $\delta_{2\bar{t}}z(x,t)$ and $\delta_{2\bar{t}}u(x,t)$ are solutions of the equations

$$\Lambda_{(A.11)}\delta_{2\bar{t}}z(x,t) = f_{(A.11)}(x,t), \qquad (x,t) \in G_h^{[2]}, \tag{A.11a}$$

$$L_{(A.12)}\delta_{2\bar{t}}u(x,t) = f_{(A.12)}(x,t), \qquad (x,t) \in G_h^{[2]}. \tag{A.12a}$$

The equations are found by applying the operator $\delta_{\bar{t}}$ to equations (A.9a), (A.10a). At the left and right boundary the following conditions are satisfied:

$$\delta_{2\bar{t}}\,z(x,t) = \varphi_{(A.11)}(x,t), \qquad (x,t) \in S_h^{[2]}, \tag{A.11b}$$

$$\delta_{2\bar{t}}u(x,t) = \varphi_{(A.12)}(x,t), \qquad (x,t) \in S_h^{[2]}, \tag{A.12b}$$

where

$$\varphi_{(A.11)}(x,t) = \varphi_{2\bar{t}}(x,t), \qquad x = 0,\ d, \qquad (x,t) \in S_h^{[2]}, \tag{A.11c}$$

$$\varphi_{(A.11)}(x,t) = \varphi^0_{(A.11)}(x) \equiv \delta_{2\bar{t}}z_{(6.3)}(x,t), \qquad t = 2\tau, \qquad (x,t) \in S_h^{[2]},$$

$$\varphi_{(A.12)}(x,t) = \varphi_{2\bar{t}}(x,t), \qquad x = 0,\ d, \qquad (x,t) \in S^{[2]}, \tag{A.12c}$$

$$\varphi_{(A.12)}(x,t) = \varphi^0_{(A.12)}(x) \equiv \delta_{2\bar{t}}\,u\,(x,t), \qquad t = 2\tau, \qquad (x,t) \in S^{[2]}.$$

First we estimate

$$\varphi^0_{(A.12)}(x) - \varphi^0_{(A.11)}(x) = \delta_{2\bar{t}}\,u(x,t) - \delta_{2\bar{t}}\,z(x,t), \qquad t = 2\tau.$$

For this purpose we write the function $u(x,t)$ in a Taylor expansion for t

$$u(x,t) = a^{(1)}(x)t + a^{(2)}(x)t^2 + v_2(x,t) \equiv u^{[2]}(x,t) + v_2(x,t), \qquad (x,t) \in \overline{G}, \tag{A.13}$$

where the coefficients $a^{(1)}(x)$, $a^{(2)}(x)$ should be determined. Inserting $u(x,t)$, in its form (A.13), into equation (2.1a) we come to the systems

$$-p(x,0)a^{(1)}(x) = f(x,0),$$

$$-2p(x,0)a^{(2)}(x) + \varepsilon^2 a\frac{\partial^2}{\partial x^2}a^{(1)}(x) - \left(c(x,0) - \frac{\partial}{\partial t}p(x,0)\right)a^{(1)}(x) = \frac{\partial}{\partial t}f(x,0),$$

from which the functions $a^{(1)}(x)$, $a^{(2)}(x)$ are found successively. The function $v_2(x,t)$ is the solution of the boundary value problem

$$L_{(2.1)}v_2(x,t) = f_{(A.14)}(x,t) \equiv f(x,t) - L_{(2.1)}u^{[2]}(x,t), \qquad (x,t)\in G, \tag{A.14}$$

$$v_2(x,t) = \varphi_{(A.14)}(x,t) \equiv \varphi(x,t) - u^{[2]}(x,t), \qquad (x,t)\in S.$$

Estimating $f_{(A.14)}(x,t)$ and $\varphi_{(A.14)}(x,t)$, and using the maximum principle we derive the estimate

$$|v_2(x,t)| \leqslant Mt^3, \qquad (x,t)\in \overline{G}. \tag{A.15}$$

Further, we have to construct the function $z(x,t)$ in the form

$$z(x,t) = (b_0^{(1)}(x) + b_1^{(1)}(x)\tau)t + b_0^{(2)}(x)t^2 + v_2^h(x,t)$$
$$\equiv z^{[2]}(x,t) + v_2^h(x,t), \qquad (x,t)\in \overline{G}_h,$$

i.e. as an expansion in powers of τ and t. Inserting $z(x,t)$ into equation (6.3), we arrive at the equations

$$-p(x,0)b_0^{(1)}(x) = f(x,0),$$

$$-2p(x,0)b_0^{(2)}(x) + \varepsilon^2 a\frac{\partial^2}{\partial x^2}b_0^{(1)}(x) - \left(c(x,0) + \frac{\partial}{\partial t}p(x,0)\right)b_0^{(1)}(x) = \frac{\partial}{\partial t}f(x,0),$$

$$b_0^{(2)}(x) + b_1^{(1)}(x) = 0.$$

So, we have

$$z^{[2]}(x,t) = u^{[2]}(x,t) + b_1^{(1)}(x)\tau t, \qquad (x,t)\in \overline{G}_h. \tag{A.16}$$

The function $v_2^h(x,t)$ is the solution of the discrete boundary value problem

$$\Lambda_{(6.3)}v_2^h(x,t) = f_{(A.17)}(x,t) \equiv f(x,t) - \Lambda_{(6.3)}z^{[2]}(x,t), \qquad (x,t)\in G_h, \tag{A.17}$$
$$v_2^h(x,t) = \varphi_{(A.17)}(x,t) \equiv \varphi(x,t) - z^{[2]}(x,t), \qquad (x,t)\in S_h.$$

Taking into account estimates of the functions $f_{(A.17)}(x,t)$ and $\varphi_{(A.17)}(x,t)$, we derive the estimate

$$\left|v_2^h(x,t)\right| \leqslant M\left[N^{-1}\ln N + t\right]t^2, \qquad (x,t)\in \overline{G}_h. \tag{A.18}$$

By virtue of relations (A.15), (A.16), (A.18) the following inequality is valid:

$$\left|\varphi^0_{(A.12)}(x) - \varphi^0_{(A.11)}(x)\right| = \left|\delta_{2\bar t}u(x,t) - \delta_{2\bar t}z(x,t)\right| \tag{A.19}$$
$$\leqslant M\left[N^{-1}\ln N + \tau\right], \qquad (x,t)\in \overline{G}_h, \qquad t = 2\tau.$$

We continue by estimating $\delta_{2\bar{t}}u(x,t) - \delta_{2\bar{t}}z(x,t)$ for $t > 2\tau$. Note that the functions $\delta_{2\bar{t}}u(x,t)$ and $\delta_{2\bar{t}}z(x,t)$ are solutions of differential and difference equations, obtained from equations (2.1) and (6.3) respectively by applying the operator $\delta_{2\bar{t}}$. Moreover, the difference equation for $\delta_{2\bar{t}}z(x,t)$ approximates the differential equation for $\delta_{2\bar{t}}u(x,t)$ ε-uniformly. On the boundary S_h, for $x = 0$ or $x = 1$, we have $\delta_{2\bar{t}}u(x,t) = \delta_{2\bar{t}}z(x,t)$. Taking into account estimate (A.19) we find

$$\left| \delta_{2\bar{t}}u(x,t) - \delta_{2\bar{t}}z(x,t) \right| \leqslant M\left[N^{-1}\ln N + \tau\right], \tag{A.20}$$

$$(x,t) \in \overline{G}_h, \qquad t \geqslant 2\tau.$$

Taking into account (6.5) we easily see

$$\left|(\Lambda_{(6.3)} - L_{(2.1)})\,u(x,t)\right| \leqslant M\left[N^{-2}\ln^2 N + \tau\right], \qquad x \neq \sigma,\ d-\sigma,$$
$$\left|(\Lambda_{(6.3)} - L_{(2.1)})\,u(x,t)\right| \leqslant M\left[\min(t,1)N^{-1}\ln N + \tau\right], \qquad x = \sigma,\ d-\sigma,$$
$$(x,t) \in G_h.$$

Proceeding in the same way as we did to obtain (A.20), we obtain the estimates

$$\left|\delta_{\bar{t}}u(x,t) - \delta_{\bar{t}}z^{(1)}(x,t)\right| \leqslant M\left[N^{-2}\ln^2 N + \tau\right], \qquad (x,t) \in \overline{G}_h, \qquad t \geqslant \tau.$$

$$\left|\delta_{2\bar{t}}u(x,t) - \delta_{2\bar{t}}z^{(1)}(x,t)\right| \leqslant M\left[N^{-2}\ln N + \tau\right], \qquad (x,t) \in \overline{G}_h, \qquad t \geqslant 2\tau. \tag{A.21}$$

$$\left|u(x,t) - z^{(2)}(x,t)\right| \leqslant M\left[N^{-2}\ln^2 N + \tau^2\right], \qquad (x,t) \in \overline{G}_h.$$

This completes the proof.

Now, as a direct consequence of the theorem, we make two remarks to prepare the proof of Theorem 4

REMARK 6 In the above we have found (A.22) for $k = 1$. In completely the same way we derive this bound for $k = 2$, so that we obtain

$$\left|\delta_{2\bar{t}}\,u(x,t) - \delta_{2\bar{t}}\,z^{(k)}(x,t)\right| \leqslant M\left[N^{-2}\ln^2 N + \tau^k\right], \tag{A.22}$$

$$(x,t) \in \overline{G}_h, \qquad t \geqslant 2\tau, \qquad k \leqslant 2.$$

REMARK 7 Making use of (A.22), similar to the derivation of estimate (A.21), we also find

$$\left|\delta_{3\bar{t}}\,u(x,t) - \delta_{3\bar{t}}\,z^{(1)}(x,t)\right| \leqslant M\left[N^{-2}\ln^2 N + \tau\right], \tag{A.23}$$

$$(x,t) \in \overline{G}_h, \qquad t \geqslant 3\tau.$$

We briefly indicate the differences with the proof given above for (A.21). To estimate the difference between $\delta_{3\bar{t}}\,u(x,t)$ and $\delta_{3\bar{t}}\,z(x,t)$ for $t = 3\tau$ we represent the function $u(x,t)$ (with condition (6.8)) in the form

$$u(x,t) = a^{(2)}(x)t^2 + a^{(3)}(x)t^3 + v_3(x,t) \equiv u^{[3]}(x,t) + v_3(x,t), \qquad (x,t) \in \overline{G},$$

and the function $z(x, t)$ in the form

$$z(x,t) = u^{[3]}(x,t) + (b_1^{(1)}(x)\tau + b_2^{(1)}(x)\tau^2)t + b_1^{(2)}(x)\tau t^2 + v_3^h(x,t)$$
$$\equiv z^{[3]}(x,t) + v_3^h(x,t), \qquad (x,t) \in \overline{G}_h.$$

The coefficients of these expansions are found using equations (2.1) and (6.3) respectively. For the coefficients we have the system

$$-p(x,0)a^{(2)}(x) = \frac{\partial}{\partial t} f(x,0),$$

$$-3p(x,0)a^{(3)}(x) + \varepsilon^2 a \frac{\partial^2}{\partial x^2} a^{(2)}(x) - \left(c(x,0) + 2\frac{\partial}{\partial t} p(x,0)\right) a^{(2)}(x) = 2^{-1}\frac{\partial}{\partial t} f(x,0),$$

$$-b_1^{(1)}(x) + a^{(2)}(x) = 0,$$

$$-2p(x,0)b_1^{(2)}(x) + \frac{\partial}{\partial t} p(x,0)a^{(2)}(x) + 3p(x,0)a^{(3)}(x)$$
$$+ \left(-\frac{\partial}{\partial t} p(x,0) - c(x,0)\right) b_1^{(1)}(x) + \varepsilon^2 a \frac{\partial^2}{\partial x^2} b_1^{(1)}(x) = 0,$$

$$-b_2^{(1)}(x) - a^{(3)}(x) + b_1^{(2)}(x) = 0.$$

The unknown functions $a^{(2)}$, $a^{(3)}$, $b_1^{(1)}$, $b_1^{(2)}$, $b_2^{(1)}$ can be found successively. For the functions $v_3(x,t)$ and $v_3^h(x,t)$ the following estimates are derived

$$|\, v_3(x,t)\,| \leqslant Mt^4, \qquad (x,t) \in \overline{G},$$
$$\left|\, v_3^h(x,t)\,\right| \leqslant M\left[N^{-2}\ln^2 N + t\right]t^3, \qquad (x,t) \in \overline{G}_h.$$

For these inequalities and the expression for $z^{[3]}(x,t)$ it follows that (A.23) holds ε-uniformly for $t = 3\tau$. The remainder of the proof of the estimate (A.23) repeats, with small variations, the proof of the estimate (A.21).

A.3 *The proof of Theorem 4*

Notice that, if for the functions $z^{(1)}(x,t)$, $z^{(2)}(x,t)$ the following relations hold

$$\left|\delta_{3\bar{t}}\, u(x,t) - \delta_{3\bar{t}}\, z^{(1)}(x,t)\right| \leqslant M\left[N^{-2}\ln^2 N + \tau\right], \qquad (x,t) \in G_h, \qquad t \geqslant 3\tau,$$
$$\left|\delta_{2\bar{t}} u(x,t) - \delta_{2\bar{t}} z^{(2)}(x,t)\right| \leqslant M\left[N^{-2}\ln^2 N + \tau^2\right], \qquad (x,t) \in G_h, \qquad t \geqslant 2\tau, \tag{A.24}$$

then for the difference $u(x,t) - z^{(3)}(x,t) \equiv \omega^{(3)}(x,t)$ we have the following

$$\left|\Lambda_{(6.3)}\omega^{(3)}(x,t)\right| \leqslant M\left[N^{-2}\ln^2 N + \tau^3\right], \qquad (x,t) \in G_h,$$

$$\omega^{(3)}(x,t) = 0, \qquad (x,t) \in S_h.$$

Hence we have

$$\left| u(x,t) - z^{(3)}(x,t) \right| \leqslant M \left[N^{-2} \ln^2 N + \tau^3 \right], \qquad (x,t) \in \overline{G}_h.$$

Thus, for the proof of the theorem it is sufficient to show inequalities (A.24). These inequalities follow from (A.22), (A.23). Thus we have proved Theorem 4.